

ON CYCLIC CAT(0) DOMAINS OF DISCONTINUITY

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ABSTRACT. Let X be a CAT(0) space, and G a discrete cyclic group of isometries of X . We investigate the domain of discontinuity for the action of G on the boundary ∂X .

1. INTRODUCTION

The idea of a domain of discontinuity was first investigated in the setting of Kleinian groups, discrete groups of isometries of hyperbolic m -space, \mathbb{H}^m . Using the Poincare ball model of \mathbb{H}^m , where \mathbb{H}^m is the interior of the open unit m -ball, we see that the boundary of \mathbb{H}^m is the unit sphere \mathbb{S}^{m-1} , and we see that isometries of \mathbb{H}^m extend to homeomorphisms of the closed m -ball, $\mathbb{H}^m \cup \mathbb{S}^{m-1}$.

Let G be a discrete group of isometries of \mathbb{H}^m , and (g_i) a sequence of distinct elements of G . Fixing a point $a \in \mathbb{H}^m$, using compactness and passing to a subsequence we may assume that $g_i(a) \rightarrow p \in \mathbb{S}^{m-1}$ and $g_i^{-1}(a) \rightarrow n \in \mathbb{S}^{m-1}$. It can be shown that $g_i(x) \rightarrow p$ uniformly on compact subsets of $\mathbb{S}^{m-1} - \{n\}$.

Definition . Let G be a group of homeomorphisms of a compact hausdorff space Z , we say that G acts on Z as a discrete convergence group if for each sequence of distinct elements of G , there is a subsequence $(g_i) \subset G$ and points $n, p \in Z$ such that $g_i(x) \rightarrow p$ uniformly on compact subsets of Z . As shown above, Kleinian groups of dimension m act as a discrete convergence group on \mathbb{S}^{m-1} .

The set of all such p is the limit set of the Kleinian group G ,

$$\Lambda(G) = \{p \in \mathbb{S}^{m-1} : \exists (g_i) \subset G \text{ with } g_i(x) \rightarrow p \text{ for some } x \in \mathbb{H}^m\}$$

and the domain of discontinuity of G , $\Omega G = \mathbb{S}^{m-1} - \Lambda G$. Clearly ΛG is G -invariant and closed, so ΩG is G -invariant and open. Let $p \in \Lambda G$ and $(g_i) \subset G$ with $g_i(x) \rightarrow p$ and $g_i^{-1}(x) \rightarrow n \in \Lambda G$ for some (any) $x \in \mathbb{H}^m$. Then for any $a \in \Omega G$, $g_i(a) \rightarrow p$. Thus if $\Omega G \neq \emptyset$ then p is a limit point of ΩG , and so ΩG is dense in \mathbb{S}^{m-1} .

We have the following result: If G is a discrete group of isometries of hyperbolic m -space, \mathbb{H}^m , then $\partial \mathbb{H}^m = \mathbb{S}^{m-1}$ is a disjoint union of the

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limit set of G , ΛG and the domain of discontinuity of G , $\Omega(G)$, where ΩG is open and either dense or empty. Since G acts as a convergence group on \mathbb{S}^{m-1} , the action of G on ΩG is properly discontinuous (hence the name domain of discontinuity).

This result holds whenever G acts as a discrete convergence group on a compact Hausdorff space, in particular when G is a discrete group of isometries of a proper δ -hyperbolic space.

In this paper we consider the case where the negative curvature (δ -hyperbolic) condition is relaxed to a non-positive curvature (CAT(0)) condition. Considering the action of \mathbb{Z}^m on \mathbb{R}^m we see that often the action on the boundary may be trivial and so the domain of discontinuity must be empty whenever the group is infinite. Thus the boundary will not in general be the union of the limit set and the domain of discontinuity.

The conjecture seems to be that if a cyclic subgroup acts “nicely” on a CAT(0) space and the action on the boundary is not virtually trivial, then there should be an open dense subset of the boundary on which the cyclic subgroup acts properly discontinuously.

This conjecture is realized when the Tits diameter of the boundary is large, but still open in the other cases. The case where the fixed points of the cyclic subgroup are spherical suspension points of the boundary will not be addressed in this note.

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2. DEFINITIONS AND BASIC RESULTS

We refer the reader to [5] or [1] for more details of the following.

Definition . For X a geodesic metric space and $\Delta(a, b, c)$ a geodesic triangle in X with vertices $a, b, c \in X$ there is a *comparison* triangle $\bar{\Delta} = \Delta(\bar{a}, \bar{b}, \bar{c}) \subset \mathbb{E}^2$ with $d(a, b) = d(\bar{a}, \bar{b})$, $d(a, c) = d(\bar{a}, \bar{c})$ and $d(b, c) = d(\bar{b}, \bar{c})$. We define the comparison angle $\bar{\angle}_a(b, c) = \angle_{\bar{a}\bar{b}\bar{a}\bar{c}}(\bar{b}, \bar{c})$.

Each point $z \in \Delta(a, b, c)$ has a unique comparison point, $\bar{z} \in \bar{\Delta}$ on the edge corresponding to the edge of z and in the corresponding location on that edge. We say that the triangle $\Delta(a, b, c)$ is CAT(0) if for any $y, z \in \Delta(a, b, c)$ with comparison points $\bar{y}, \bar{z} \in \bar{\Delta}$, $d(y, z) \leq d(\bar{y}, \bar{z})$. The space X is said to be CAT(0) if every geodesic triangle in X is CAT(0).

If X is CAT(0), notice that for any geodesics $\alpha : [0, r] \rightarrow X$ and $\beta : [0, s] \rightarrow X$ with $\alpha(0) = \beta(0) = a$, the function $\theta(r, s) = \angle_{r(0)}(\alpha(r), \beta(s))$ is an increasing function of r, s . Thus $\lim_{r, s \rightarrow 0} \theta(r, s)$ exists and we call this

limit $\angle_a(\alpha(r), \beta(s))$. It follows that for any $a, b, c \in X$, a CAT(0) space, $\angle_a(b, c) \leq \bar{\angle}_a(b, c)$.

The (visual) boundary, ∂X , is the set of equivalence classes of rays, where rays are equivalent if they fellow travel. Given a ray R and a point $x \in X$ there is a ray S emanating from x with $R \sim S$. Fixing a base point $\mathbf{0} \in X$ we defined a Topology on $\bar{X} = X \cup \partial X$ by taking the basic open sets of $x \in X$ to be the open metric balls about x . For $y \in \partial X$, and R a ray from $\mathbf{0}$ representing y , we construct basic open sets $U(R, n, \epsilon)$ where $n, \epsilon > 0$. We say $z \in U(R, n, \epsilon)$ if the unit speed geodesic, $S : [0, d(\mathbf{0}, z)] \rightarrow \bar{X}$, from $\mathbf{0}$ to z satisfies $d(R(n), S(n)) < \epsilon$. These sets form a basis for a regular topology on \bar{X} and ∂X . For any $x \in X$ and $u, v \in \partial X$ we can define $\angle_x(u, v)$ and $\bar{\angle}_x(u, v)$ by parameterizing the rays $[x, u)$ and $[x, v)$ by $t \in [0, \infty)$ and taking the limit as $t \rightarrow 0$ and $t \rightarrow \infty$ respectively.

for $u, v \in \partial X$, we define $\angle(u, v) = \sup_{p \in X} \angle_p(u, v)$. It follows from [5]

that $\angle(u, v) = \bar{\angle}_p(u, v)$ for any $p \in X$. Notice that isometries of X preserve the angle between points of ∂X . The angle defines a path metric, d_T on the set ∂X , called the Tits metric, whose topology is finer than the given topology of ∂X . Also $\angle(a, b)$ and $d_T(a, b)$ are equal whenever either of them are less than π .

The set ∂X with the Tits metric is called the Tits boundary of X , denoted TX . Isometries of X extend to isometries of TX .

The identity function $TX \rightarrow \partial X$ is continuous, but the identity function $\partial X \rightarrow TX$ is only lower semi-continuous. That is for any sequences $(u_n), (v_n) \subset \partial X$ with $u_n \rightarrow u$ and $v_n \rightarrow v$ in ∂X , then

$$\liminf d_T(u_n, v_n) \geq d_T(u, v)$$

Lemma 1. *Let $v \in \partial X$ and $H \subset X$ a horoball centered at v , then $\bar{H} \cap \partial X \subset \bar{B}_T(v, \frac{\pi}{2})$ the closed Tits ball about v .*

Proof. Suppose that $w \in \bar{H} \cap \partial X$ with $\angle(v, w) > \frac{\pi}{2}$. Choose a point $y \in X$ with $\angle_y(v, w) > \frac{\pi}{2}$.

Let $R : [0, \infty) \rightarrow X$ be the geodesic ray from y to v , and $b_R : X \rightarrow \mathbb{R}$ be the Buseman function associated to R , $b_R(x) = \lim_{t \rightarrow \infty} [d(x, R(t)) - t]$. Note the $b_R(y) = 0$ and $b_R(R(t)) = -t$ for $t > 0$. By [5, Exercise II 8.23(1)] any horoball based at v is within finite distance of any other horoball based at v . It follows that all horoballs based at v will have the same limit points in the boundary. Thus we may assume that $H = b_R^{-1}([-1, -\infty))$.

Since $w \in \bar{H}$, there exists $\hat{w} \in H$ such that $\angle_y(v, \hat{w}) > \frac{\pi}{2}$.

We recall [5, Exercise II 8.23(4)]. Choose a geodesic line $\bar{R} : \mathbb{R} \rightarrow \mathbb{E}^2$. For each $x \in X$ and $n \in \mathbb{N}$ consider a comparison triangle $\bar{\Delta}(x, R(0), R(n))$ with $\bar{R}(i)$ the comparison point for $R(i)$ ($i = 0, n$) and $x_n \in \mathbb{E}^2$ the comparison point for x . Choose $r_n \in \mathbb{R}$ with $\bar{R}(r_n) = \pi_{\bar{R}}(x_n)$ (where $\pi_{\bar{R}}$ is orthogonal projection to the line \bar{R}). Then $r_n \rightarrow r$ where $b_R(x) = -r$. Since $b_R(\hat{w}) \leq -1$, for $n \gg 0$, $r_n > 0$. It follows that $\angle_{\bar{R}(0)}(\bar{R}(n), \hat{w}_n) < \frac{\pi}{2}$. However

$$\frac{\pi}{2} < \angle_y(v, \hat{w}) = \angle_{R(0)}(R(n), \hat{w}) \leq \bar{\angle}_{R(0)}(R(n), \hat{w}) = \angle_{\bar{R}(0)}(\bar{R}(n), \hat{w}_n) < \frac{\pi}{2}$$

which is a contradiction. \square

3. LARGE TITS RADIUS

Recall that for any $u \in \partial X$, $B_T(u, \epsilon) = \{v \in \partial X : d_T(u, v) < \epsilon\}$ and $\bar{B}_T(u, \epsilon) = \{v \in \partial X : d_T(u, v) \leq \epsilon\}$, where d_T is the Tits metric. For $A \subset \partial X$, we define

$$\text{radius}_T(A) = \inf\{r : A \subset B_T(u, r) \text{ for some } u \in \partial X\}.$$

If g is an infinite order isometry of X , and $\langle g \rangle$ is proper, then g is either hyperbolic or parabolic. When g is hyperbolic it acts by translation on a line (called an axis of g) in X with endpoints g^+ (in the direction of translation) and g^- (see [5]).

Recall that a hyperbolic isometry h of X is called *rank 1*, if h has an axis L which doesn't bound a half flat. From [1] we see that $d_T(h^+, \alpha) = \infty$ for all $\alpha \neq h^+$, so if X has a rank 1 isometry then $\text{radius}_T(\partial X) = \infty$.

Recall a result from [11]

Theorem 2. [11] *Let X be a complete $\text{CAT}(0)$ space and (g_i) a sequence of isometries of X with the property that $g_i(x) \rightarrow p \in \partial X$ and $g_i^{-1}(x) \rightarrow n \in \partial X$ for any $x \in X$. Then for any $\theta \in [0, \pi]$ and any compact set $K \subset \partial X - \bar{B}_T(n, \theta)$, $g_n(K) \rightarrow \bar{B}_T(p, \pi - \theta)$ (in the sense that for any open $U \supset \bar{B}_T(p, \pi - \theta)$, $g_n(K) \subset U$ for all n sufficiently large).*

This Theorem is stated in [11] with stronger hypothesis, but they were not used in the proof. We will refer to this result as π -convergence. For a given hyperbolic element h with axis L , we set $g_i = h^i$, $n = L(-\infty) = h^-$ and $p = L(\infty) = h^+$.

The following theorem is due to Ballmann.

Theorem 3 (Ballmann). *If h is a rank 1 isometry of the complete $\text{CAT}(0)$ space X , then the group generated by h , $\langle h \rangle$ act properly discontinuously on the open set $\partial X - \{h^\pm\}$ which is dense if it is non-empty.*

Proof. Let K be a compact subset of $\partial X - \{h^\pm\}$. The Tits distance from K to $n = h^-$ will be infinite. Thus by π -convergence $h^i(K) \rightarrow p = h^+$ and $h^{-i}(K) \rightarrow n = h^-$. Thus $\{i \in \mathbb{Z} : h^i(K) \cap K \neq \emptyset\}$ is finite and the action of $\langle h \rangle$ on $\partial X - \{h^\pm\}$ is properly discontinuous. Clearly $\partial X - \{h^\pm\}$ is an open subset and for any $a \in \partial X - \{h^\pm\}$, $h^i(a) \rightarrow h^+$ and $h^{-i}(a) \rightarrow h^-$ so $\partial X - \{h^\pm\}$ is dense in ∂X . \square

Definition . For X a complete CAT(0) space we define the limit set $\Lambda X \subset \partial X$ to be the set $\{p \in \partial X : \exists (g_i) \text{ isometries of } X \text{ with } g_i(x) \rightarrow p \text{ and } g_i^{-1}(x) \rightarrow n \text{ for some } n \in \partial X \text{ and for all } x \in X\}$

The following is a slight generalization of a result of Karlsson [8].

Theorem 4. *Let X be a complete CAT(0) space with $\text{radius}_T(\partial X) > 3\pi$ and $|\partial X| > 2$. If h is a hyperbolic isometry of X then $\langle h \rangle$ acts properly discontinuously on the open subset $\Omega = \partial X - [\bar{B}_T(h^+, \frac{\pi}{2}) \cup \bar{B}_T(h^-, \frac{\pi}{2})]$. If $[\bar{B}_T(h^+, \frac{\pi}{2}) \cup \bar{B}_T(h^-, \frac{\pi}{2})] \subset \Lambda X$ then Ω is dense in ∂X .*

Proof. By Theorem 3 we may assume h is not rank 1. Thus some axis of h bounds a half-flat, which corresponds to a Tits geodesic of length π from h^+ to h^- so $d_T(h^+, h^-) = \pi$.

If there is any point of $w \in \partial X$ which is isolated in the Tits metric, ($d_T(w, v) = \infty, \forall v$) then by π -convergence the orbit of w under $\langle h \rangle$ is infinite and each element of the orbit will be isolated as well.

By the triangle inequality, for any $q \in \partial X$, there exists $u, v \in \partial X$ such that $d_T(q, u), d_T(q, v) \geq \pi$ and $d_T(u, v) > 2\pi$. It follows from π -convergence that for any $w \in \Lambda X$ and any neighborhood W of w in ∂X , the Tits diameter $\text{diam}_T(W) > 2\pi$.

Since $\text{diam}_T[\bar{B}_T(h^+, \frac{\pi}{2}) \cup \bar{B}_T(h^-, \frac{\pi}{2})] \leq 2\pi$, $W \not\subset \bar{B}_T(h^+, \frac{\pi}{2}) \cup \bar{B}_T(h^-, \frac{\pi}{2})$ so $W \cap \Omega \neq \emptyset$. Thus Ω is dense in ΛX . Since closed Tits balls are closed in ∂X , Ω is open.

For any compact $K \subset \Omega$, by π -convergence $h^i(K) \rightarrow \bar{B}_T(h^+, \frac{\pi}{2})$ and $h^{-i}(K) \rightarrow \bar{B}_T(h^-, \frac{\pi}{2})$. Using this, we can show that $\{i \in \mathbb{Z} : h^i(K) \cap K \neq \emptyset\}$ is finite and so the action of $\langle h \rangle$ on Ω is properly discontinuous. \square

Example 5. Notice that Ω need not be dense in ∂X if we remove the condition that $\bar{B}_T(h^\infty, \frac{\pi}{2}) \cup \bar{B}_T(h^{-\infty}, \frac{\pi}{2}) \subset \Lambda X$. You start with the half plane $\{(x, y) : y \geq 0\}$ where h is unit translation in the 1st coordinate. Now you attach a line at the origin and let g act by translation in that line. You construct the CAT(0) space X by translating this picture by g and h . This gives you an action of $\langle g, h \rangle$ on the CAT(0) space X . The isometry g is a rank 1 hyperbolic element, so X is rank 1, but $\Omega = \partial X - [\bar{B}_T(h^\infty, \frac{\pi}{2}) \cup \bar{B}_T(h^{-\infty}, \frac{\pi}{2})]$ is not dense in the boundary, in fact its closure hits $[\bar{B}_T(h^\infty, \frac{\pi}{2}) \cup \bar{B}_T(h^{-\infty}, \frac{\pi}{2})]$ only in $h^{\pm\infty}$.

Theorem 6. *Let X be a proper CAT(0) space with $\text{radius}_T(\partial X) > 3\pi$, and h a parabolic isometry of X . There exists $m \in \partial X$ fixed point of h such that $\langle h \rangle$ acts properly discontinuously on the open dense subset $\Omega = \partial X - \bar{B}_T(m, \pi)$ of ∂X . If $\bar{B}_T(m, \pi) \subset \Lambda X$ then Ω is dense in ∂X .*

Proof. Clearly Ω is non-empty and open. By [5, II 8.25], there exists $m \in \partial X$ such that h leaves invariant each horoball centered at m which implies that h fixes m . Suppose that $\langle h \rangle$ doesn't act properly discontinuously on Ω . Then there exists K , a compact subset of Ω , and a strictly increasing sequence $(i_j) \subset \mathbb{N}$ such that $K \cap h^{i_j}(K) \neq \emptyset$ for all $j \in \mathbb{N}$.

Fix $x \in X$. Passing to a subsequence we may assume that $h^{i_j}(x) \rightarrow p \in \partial X$ and $h^{-i_j}(x) \rightarrow n \in \partial X$. Since h leaves the horosphere S centered at m passing through x invariant, then $n, p \in \partial S \subset \bar{B}_T(m, \frac{\pi}{2})$ by Lemma 1. Since ∂X is Hausdorff, there exists $U \subset \partial X$ open with $\bar{B}_T(m, \pi) \subset U$ and $U \cap K = \emptyset$. Since $\bar{B}_T(n, \frac{\pi}{2}), \bar{B}_T(p, \frac{\pi}{2}) \subset \bar{B}_T(m, \pi) \subset U$, by π -convergence $h^{i_j}(K) \subset U$ for all $j \gg 0$ which implies $h^{i_j}(K) \cap K = \emptyset$ for all $j \gg 0$ contradicting the choice of (i_j) . Thus $\langle h \rangle$ acts properly on Ω .

By the triangle inequality, for any $q \in \partial X$, there exists $u, v \in \partial X$ such that $d_T(q, u), d_T(q, v) \geq \pi$ and $d_T(u, v) > 2\pi$. It follows from π -convergence that for any $w \in \Lambda X$ and any neighborhood W of w in ∂X , the Tits diameter $\text{diam}_T(W) > 2\pi$.

Since $\text{diam}_T(\bar{B}_T(m, \pi)) \leq 2\pi$, $W \not\subset \bar{B}_T(m, \pi)$ so $W \cap \Omega \neq \emptyset$. Thus Ω is dense in ΛX . \square

4. SMALL TITS RADIUS

Definition . Recall that a metric space is *proper* if closed metric balls are compact, and a metric space is *cocompact* if the quotient of the space by the isometry group is compact.

We now assume that X is a proper CAT(0) space.

Definition . We say that $a, b \in \partial X$ are antipodes if $d_T(a, b) \geq \pi$. The suspension of the antipodes a and b , S_a^b , is the union of a and b together with of all Tits geodesics from a to b of length π (if any). Notice that $S_a^b = S_b^a$.

Let $a \in \partial X$ and (g_i) a sequence of isometries of a CAT(0) space X . We say that (g_i) *pulls from* n if there is some unit speed geodesic ray $R : [0, \infty) \rightarrow X$ representing n , and a sequence $(s_i) \subset [0, \infty)$ with $s_i \rightarrow \infty$ such that the sequence $(g_i(R(s_i)))$ is bounded. Clearly this is independent of the ray chosen. Passing to a subsequence, we may assume that $g_i(R(s_i)) \rightarrow b$, and that the sequence of rays $(g_i(R))$ (each

reparametrized) converges uniformly on compact subsets to a geodesic line L with $g_i(-i) \rightarrow L(-\infty)$, and $L(0) = b$. Notice that for some (and so any) $x \in X$, $g_i(x) \rightarrow L(-\infty)$ and $g_i^{-1}(x) \rightarrow n$.

For any $x \in \partial X$, passing to a subsequence, we may assume that $g_i(x) \rightarrow \hat{x} \in \partial X$. Thus for any compact set $C \subset TX$ (where TX is ∂X with the topology of the Tits metric) we can define $f : C \rightarrow \partial X$ by $f(x) = \hat{x}$. In fact we can define f whenever C has a countable dense subset as a subset of TX . By [5] $f : C \rightarrow TX$ is Lipschitz with constant one. Notice that $f(n) = L(-\infty)$.

Lemma 7. *In the above setting for any $a \in \bar{B}_T(n, \pi)$, $f : [n, a] \rightarrow TX$ is an isometric embedding of $[n, a]$ into $S_{L(-\infty)}^{L(\infty)}$.*

Proof. Because f is Lipschitz with constant one, if $d_T(n, a) = d_T(f(n), f(a))$ then $f : [n, a] \rightarrow TX$ is an isometry. Since $f(n) = L(-\infty)$, it suffices to show that $d_T(f(n), f(a)) + d_T(f(a), L(\infty)) = \pi$.

Let $\theta = d_T(n, a)$, so by π -convergence $\pi - \theta \geq d_T(f(a), L(\infty))$. Since f is Lipschitz with constant one, $d_T(f(n), f(a)) \leq \theta$. However

$$\pi = \theta + (\pi - \theta) \geq d_T(f(n), f(a)) + d_T(f(a), L(\infty)) \geq d_T(L(-\infty), L(\infty)) \geq \pi$$

□

Example 8. The function f defined above need not be an embedding on $B_T(n, \pi)$. Consider the half flat $Y = \mathbb{R} \times [0, \infty)$. For each $n \in \mathbb{Z}$ glue the quarter flat $X_n = [n, \infty) \times [0, \infty)$ by identifying $(a, b) \in Y$ with $(c, d) \in X_n$ if $a = c$, $b = d$, and $\ln(a + 1 - n) \geq b$. Let Z be the resulting space. ∂Z is broom attached at ∞ in the \mathbb{R} factor of Y . Notice that \mathbb{Z} acts on Z ($1 \in \mathbb{Z}$ sends X_n to X_{n+1}). Letting g_i be subtraction by i , we have that $f(\partial Z) = \partial Y$.

Definition . Let g be a hyperbolic isometry of the CAT(0) space X . We define $S_g = S_{g^-}^+$, that is the suspension of the endpoints of g .

Lemma 9. *The group $\langle g \rangle$ acts discretely on the open set $\Omega = \partial X - S_g$.*

Proof. It suffices to show that for any point $a \in \partial X$, the limit points of the $\{g^i(a) : i \in \mathbb{N}\}$ are in S_g . Let b be a limit point of $\{g^i(a) : i \in \mathbb{N}\}$. Choose an increasing sequence $(i_j) \subset \mathbb{N}$ such that $g^{i_j}(a) \rightarrow b$.

Let L be an axis of g , then the sequence (g^{i_j}) pulls from $n = L(-\infty) = g^-$, leaving L invariant. If $d_T(a, n) \geq \pi$ then by π -convergence $b = L(\infty) = g^+$. If $d_T(a, n) < \pi$ then by we construct the function $f : [n, a] \rightarrow TX$ as above and so $f(a) = b$.

By Lemma 7 f embeds $[n, a]$ into $S_{L(-\infty)}^{L(\infty)} = S_g$. Thus $b \in S_g$. □

Definition . An bounded metric space X is almost a product(K) of Y and Z if there is a convex subset $W \cong Y \times Z$ and $X \subset \text{Nbh}(W, K)$.

Lemma 10. *Let X be a proper cocompact $\text{CAT}(0)$ space X . If there is a convex subset W of X where $W = Y \times Z$, and a non-empty open subset $V \subset \partial X$ with $V \subset \partial W$ then X is almost a product of \hat{Y} and \hat{Z} , convex subsets of X . Additionally if Y is cocompact then $\hat{Y} \cong Y$.*

Proof. By [7] X has almost extendable(C) geodesics for some C . That is for any $x, y \in X$ there exists $z \in B(y, C)$ such that the geodesic segment $[x, z]$ extends to a ray from x .

We may assume that Y and Z are (convex) subsets of W . Fix a base point $w \in W$. Let $\alpha \in V$. Let $R : [0, \infty) \rightarrow W$ be a geodesic ray with $R(0) = w$ and $R(\infty) = \alpha$. There exists $N \in \mathbb{N}$, $\epsilon > 0$ such that $U(\alpha, N, \epsilon) \subset V$.

Using cocompactness, we find $(g_i) \subset G$ and $t_i \rightarrow \infty$ such that $g_i(R_{t_i}) \rightarrow L$ uniformly on compact subset where R_{t_i} is the ray R reparametrize to have domain $[-t_i, \infty)$. Using the fact that X is proper and passing to a subsequence, there are unbounded sequences $(y_i) \subset Y$, $(z_i) \subset Z$ so that $g_i(w_i) \rightarrow \hat{w} \in X$ where $w_i = (y_i, z_i)$. Using properness and passing to a subsequence we have that $g_i(W) \rightarrow \hat{W}$, a convex subset of X . Let Y_i be the copy of Y in W which contains z_i , and Z_i the copy of Z in W which contains y_i , so that $Y_i \cap Z_i = \{w_i\}$. Passing to a subsequence, we may assume that $g_i(Y_i) \rightarrow \hat{Y}$, a convex subset of \hat{W} and $g_i(Z_i) \rightarrow \hat{Z}$, a convex subset of \hat{W} . Now let $\hat{a}, \hat{b} \in \hat{W}$. Thus there are sequences $(a_i), (b_i) \subset W$ with $g_i(a_i) \rightarrow \hat{a}$ and $g_i(b_i) \rightarrow \hat{b}$. For each i , let $a_i^Y = \pi_{Y_i}(a_i)$, the projection of a_i in Y_i . Define b_i^Y, a_i^Z and b_i^Z similarly. For each i , W is a product of Y_i and Z_i so $d(a_i, b_i)^2 = d(a_i^Y, b_i^Y)^2 + d(a_i^Z, b_i^Z)^2$. The sequence $(g_i(a_i^Y))$ will clearly converge to the closest point projection $\pi_{\hat{Y}}(\hat{a})$. Similarly $g_i(a_i^Z) \rightarrow \pi_{\hat{Z}}(\hat{a})$, $g_i(b_i^Y) \rightarrow \pi_{\hat{Y}}(\hat{b})$, and $g_i(b_i^Z) \rightarrow \pi_{\hat{Z}}(\hat{b})$. It follows that

$$d(\hat{a}, \hat{b})^2 = d(\pi_{\hat{Y}}(\hat{a}), \pi_{\hat{Y}}(\hat{b}))^2 + d(\pi_{\hat{Z}}(\hat{a}), \pi_{\hat{Z}}(\hat{b}))^2$$

so $\hat{W} = \hat{Y} \times \hat{Z}$.

When Y admits a cocompact action we can arrange it so that for some $y \in Y$, and h_i isometry of Y , $g_i(h_i(y))$ is bounded and so there is an isometric embedding $h : Y \rightarrow X$ such that $h(Y) = \hat{Y}$, and similarly for Z .

Now we claim that $X \subset \text{Nbh}(\hat{W}, C)$. Suppose not, then let $x \in X$ with $d(x, \hat{W}) > C$. Notice that $g_i^{-1}(\bar{B}(x, C)) \rightarrow \alpha$. Thus for $i \gg 0$ $g_i^{-1}(\bar{B}(x, C)) \subset \tilde{U}(\alpha, N, \epsilon)$.

By almost extendable geodesics(C) there is a point $v_i \in g_i^{-1}(\bar{B}(x, C))$ such that v_i is on a ray $S : [0, \infty) \rightarrow X$ from our base point w . Notice that for $i \gg 0$, $d(x, g_i(W)) > C$, so in that case $d(v_i, W) > 0$, so $S([0, \infty)) \not\subset W$.

Since the (unit speed geodesic) rays based at w give unique representatives of ∂X and every element of ∂W is represented by a ray in W from w , this implies that $S(\infty) \notin \partial W$, but $S(\infty) \in U(\alpha, N, \epsilon) \subset \partial W$ which is a contradiction. \square

Corollary 11. *Let X be a proper cocompact CAT(0) space and W a convex subset with cocompact stabilizer in the isometry group of X . If for some nonempty open $V \subset \partial X$, $V \subset \partial W$, then X lies in a uniform neighborhood of W .*

Proof. In the proof of Lemma 10, we may choose the sequence (g_i) in the stabilizer of W , so $\hat{W} = W$. \square

Unfortunately, the previous result raises more questions than it answers

Question . If X and Y are cocompact proper CAT(0) spaces and $U \subset \partial X$, $V \subset \partial Y$ non-empty with $U \cong V$, is $\partial X \cong \partial Y$? (where \cong is a homeomorphism in cone topology and an isometry in the Tits metric)

What if additionally we have a group G acting geometrically on both X and Y ? The answer is clearly yes if X or Y is rank 1 and the isomorphism $U \cong V$ is G -equivariant.

See [6] and [10] for related questions.

We now generalize part of a result of Lytchak to our setting.

Theorem 12. [9] *Let Z be a geodesically complete finite dimensional CAT(1) space. Then Z has a unique decomposition $Z = \mathbb{S}^n * G_1 * \cdots * G_k * Y_1 * \cdots * Y_m$ where G_j is a thick irreducible building and Y_j is an irreducible (via spherical join) non-building.*

In our setting, the Tits boundary TX is not geodesically complete. Nonetheless, we obtain the following result: If Z is a finite dimensional CAT(1) space then there is a unique decomposition $Z = S^n * Y$ where Y doesn't have a sphere as a spherical join factor.

Lemma 13. *If Y and Z are CAT(0) spaces and F is a flat sector in $Y \times Z$, then $\pi_Y(F)$ is a flat sector in Y .*

Proof. Since a flat sector is the nested union of flat triangles (and this is sufficient). It suffices to show that the projection of a flat triangle is a flat triangle.

Let abc be a triangle in a CAT(0) space. Consider the following condition: For $t \in (0, 1)$ if $e \in [a, b]$ and $f \in [a, c]$ with $d(a, e) = td(a, b)$ and $d(a, f) = td(a, c)$ then $d(e, f) = td(b, c)$. Notice that if abc is flat then this condition is satisfied by similar triangles. If on the other hand this condition is satisfied, then by similar triangles, the Euclidian comparison angle $\bar{\angle}_a(b, c) = \bar{\angle}_a(e, f)$. It follows that $\angle_a(b, c) = \angle_a(b, c)$ and so the triangle abc is flat.

Thus we may assume that abc is a triangle in satisfying this condition. For $t \in (0, 1)$ choose e and f as above. Let a_1, b_1, c_1, e_1, f_1 be the projections of a, b, c, e, f respectively into Y and a_2, b_2, c_2, e_2, f_2 the projections of a, b, c, e, f respectively into Z . By similar Euclidean triangles $d(a_i, e_i) = td(a_i, b_i)$ and $d(a_i, f_i) = td(a_i, c_i)$ for $i = 1, 2$.

By hypothesis $d(e, f) = td(b, c)$ so $d(e, f)^2 = t^2 d(b, c)^2$. Thus

$$d(e_1, f_1)^2 + d(e_2, f_2)^2 = t^2 d(b_1, c_1)^2 + t^2 d(b_2, c_2)^2$$

By the CAT(0) inequality applied to the triangles $a_i b_i c_i$, $i = 1, 2$. $d(e_i, f_i) \leq td(b_i, c_i)$ for $i = 1, 2$. It follows that $d(e_i, f_i) = td(b_i, c_i)$ for $i = 1, 2$. and so the triangle $a_i b_i c_i$ are flat for $i = 1, 2$. \square

Definition . [5, I 5.11] For Y and Z metric spaces with metrics bounded by π , we define $Y * Z$ to be the quotient of the space $Y \times Z \times [0, \frac{\pi}{2}]$ by the identifications $(y, z, 0) = (y, \hat{z}, 0)$ and $(y, z, \frac{\pi}{2}) = (\hat{y}, z, \frac{\pi}{2})$. We denote the class (y, z, θ) by $y \cos \theta + z \sin \theta$ so $(y, z, 0) = y$ and $(y, z, \frac{\pi}{2}) = z$, and so we have $Y, Z \subset Y * Z$. For $u = y \cos \theta + z \sin \theta$, $u' = y' \cos \theta' + z' \sin \theta' \in Y * Z$ we define $d(u, u')$ by

$$\cos(d(u, u')) = \cos \theta \cos \theta' \cos(d(y, y')) + \sin \theta \sin \theta' \cos(d(z, z'))$$

Clearly for $\theta \neq 0, \frac{\pi}{2}$, $d(u, u') = \pi$ if and only if $d(y, y') = \pi = d(z, z')$ and $\theta + \theta' = \frac{\pi}{2}$.

Lemma 14. *Let α be a geodesic in $A * B$ where A and B are CAT(1) spaces. Notice that A and B are π -convex subsets of $A * B$. If the image of α misses B , then the projection of α to A is a geodesic.*

Proof. Let $Y = C_0(A)$ and $Z = C_0(B)$ be the Euclidian cones on A and B respectively. By [5] $C_0(A * B) = X \cong Y \times Z$ and so $A * B = \partial(Y \times Z)$. By [5, I 5.15] It suffices to show that the projection of α to A is a local Tits geodesic. There is a flat sector $F \subset X$ based at the cone point 0 of X with $\partial F = \alpha$. We can think of Y and Z as being convex subsets of X . By Lemma 13, the projection F_Y of F into Y is a Euclidean sector based at the cone point of Y (which is 0 of course). By the proof of [5, I 5.15] the ∂F_Y will be the projection of α to A . It follows that this projection is a Tits geodesic in A . \square

Theorem 15. *If Z is a finite dimensional complete CAT(1) space, then there is a unique decomposition $Z = S * D$ where $S \cong \mathbb{S}^n$ and D doesn't have a non-empty sphere as a spherical join factor.*

Proof. Suppose that $Z = \mathbb{S}^k * U$ and $Z = \{a, b\} * A$, where $\{a, b\} \not\subset \mathbb{S}^k$. It follows that a is the unique antipode to b in Z and visa versa (so $\{a, b\} \cap \mathbb{S}^k = \emptyset$). By the metric on $Z = \mathbb{S}^k * U$, we have $a_1, b_1 \in \mathbb{S}^k$, $a_2, b_2 \in U$ and $\theta \in (0, \pi/2]$ with $a = a_1 \cos \theta + a_2 \sin \theta$ and $b = b_1 \sin \theta + b_2 \cos \theta$. It follows that a_2 is the unique antipode of b_2 in U and visa versa (and similarly for a_1 and b_1 in \mathbb{S}^k).

Since $Z = \{a, b\} * A$ then $Z = S_a^b$, and every point of Z is on a Tits geodesic from a to b . Fix $u \in U \subset \mathbb{S}^k * U$; the point u is on a Tits geodesic from a to b which misses \mathbb{S}^k . By Lemma 13, the projection of this Tits geodesic into U is a Tits geodesic from a_2 to b_2 passing through u . It follows that U is the union of all Tits geodesics in U from a_2 to b_2 . This implies by [9, 4.1] that $U = \{a_2, b_2\} * W$ where W is the set of all points of U at Tits distance $\pi/2$ from both a_2 and b_2 . Thus by [5, I 5.15], $Z = \mathbb{S}^k * (\{a_2, b_2\} * W) = (\mathbb{S}^k * \{a_2, b_2\}) * W = \mathbb{S}^{k+1} * W$. Since Z is finite dimensional this process must terminate, and so for some n , $Z = \mathbb{S}^n * Y$ where Y doesn't have a non-empty sphere as a spherical join factor. Notice by our construction that

$$S = \{a \in Z : Z \text{ is a spherical suspension with suspension point } a\}$$

Thus this decomposition is canonical. \square

Corollary 16. *Under the hypothesis of Lemma 10, if $Y \cong \mathbb{E}^n$ for some n then $\partial Y \subset S$, the set of suspension points of ∂X .*

Proof. By [13] ∂X is finite dimension, so TX is finite dimensional, and $\partial X = S * D$ as in Theorem 15. Since \mathbb{E}^n admits a cocompact isometric group action, $\hat{Y} \cong Y \cong \mathbb{E}^n$. $\partial X = \partial(\hat{Y} \times \hat{Z}) = (\partial Y) * (\partial \hat{Z})$. Since $\partial \hat{Y} \cong \partial \mathbb{E}^n = S^{n-1}$ then $\partial \hat{Y} \subset S$. Let $\alpha \in \partial Y - S$, so $d_T(\alpha, D) < \pi/2$. We may assume $g_i(\alpha) \rightarrow \beta$ preforce with $\beta \in \partial \hat{Y} \subset S$. However $g_i(D) = D$ for all i , so $d_T(\beta, D) \leq d_T(\alpha, D) < \pi/2$, but that is a contradiction since every point of S is distance $\pi/2$ from every point of D . Thus $\partial Y \subset S$. \square

Theorem 17. *Let g be a hyperbolic isometry of the cocompact CAT(0) space X . If $\Omega = \partial X - S_g \neq \emptyset$, then Ω is a dense open subset of ∂X , and $\langle g \rangle$ acts discretely on Ω .*

Proof. By Lemma 9, $\langle g \rangle$ acts discretely on the open set Ω . We must show that Ω is dense whenever it is non-empty.

Suppose that Ω is not dense. Then there is an open subset V of ∂X with $V \subset \partial S_g$. Recall that $S_g \cong \mathbb{R} \times Z$ for some Z , where the endpoints

of \mathbb{R} are g^\pm . Thus by Corollary 16, $g^\pm \in S$, the set of suspension points of ∂X . It follows that $S_g = \partial X$ as required. \square

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